

# Spatial correlations of singularity strengths in multifractal branching processes

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(12.12.1997 chaos-dyn/9804028)

The  $n$ -point statistics of singularity strength variables for multiplicative branching processes is calculated from an analytic expression of the corresponding multivariate generating function. The key ingredient is a branching generating function which can be understood as a natural generalisation of the multifractal mass exponents. Various random multiplicative cascade processes pertaining to fully developed turbulence are discussed.

47.27.Eq, 02.50.Sk, 05.40.+j

## I. INTRODUCTION

Many complex processes occurring in nature are believed to be organized in a selfsimilar way: turbulence, large-scale structure formation in the universe, high-energetic multiparticle dynamics and diffusion limited aggregation are but a few examples showing scaling over a wide range.

A convenient tool to characterize the spatial configurations of these processes is the multifractal formalism [1]–[3] which analyses the one-point statistics of singularity strengths under the assumption of multiple local scale invariance. Typically, scale-invariance is a consequence of some hierarchical organisation of the underlying process. Since this process occurs in space, its characterisation in terms of multifractals can be expected also to yield information about spatial correlation functions.

Multivariate correlations, on the other hand, have long been known to contain (at least in principle) complete information of the underlying random process. Moments and especially cumulants are capable, in their multivariate form, of providing this information by means of well-tested statistical procedures. As in the case of multifractals, cumulants can detect and characterize scale invariance, but unlike the former do not depend on its presence.

The existence of these two sets of tools naturally leads to the question how they are interrelated: given a selfsimilar process, does the multifractal characterisation completely determine the spatial correlation functions and vice versa, or does one provide more information than the other?

Some attempts have been made to elucidate the relationship between multifractals and correlation functions: two-point statistics of multifractal measures have been discussed in Refs. [4,5] and applied to turbulence in [6]. It is to be expected, though, that only the full  $n$ -point statistics of singularity strengths would provide the equivalent information contained in the spatial correlation functions of all orders.

It is known that different selfsimilar processes can lead to identical multifractal exponents [7–9]. This can be understood as a hint that the multifractal characterisation is indeed incomplete and that the spatial correlation functions may contain more information about the underlying process. We shall show that this is indeed the case for several examples of multiplicative branching processes.

The ultimate goal for any analysis, of course, would be a characterisation of the multivariate statistics gained from a physical process to a degree of completeness that would permit precise reconstruction of the “branching rule” that governs each step of the selfsimilar process. As far as theory is concerned, this holy grail would be achieved with the specification of a multivariate generating function whose  $n$ -fold derivatives would precisely reproduce the corresponding experimentally measured  $n$ -variate cumulants.

Clearly, finding this real-life generating function remains beyond present-day capabilities, for both theoretical and experimental reasons. It is possible, though, to make progress towards the ideal by inventing simple branching models whose generating functions can be calculated. An approach to calculate the  $n$ -point spatial correlations to arbitrary order within some of these models was presented in Refs. [10,11]: there, the multivariate generating function of the spatial correlations was constructed iteratively from a backward evolution equation, leading to a recursive derivation of spatial correlations. An important further step has been the recent discovery of a large class of analytic solutions for generating functions of selfsimilar multiplicative branching processes [12].

In this paper, we expand on the latter discovery and attempt to cast further light on said relationship between multifractals and correlations. After a brief explanation of the mechanics of multiplicative branching processes in Sec. II, we review in Sec. III the formalism of multivariate correlations and derive the abovementioned analytic solutions for the branching generating functions. While singularity strength variables are already treated in Sec. III, the relationship between spatial correlations and multifractals is explored more fully in Sec. IV, where we also show how the vaunted splitting function (“branching rule”) can be reconstructed via a two-dimensional Laplace transform. The question of reconstructing this splitting function from data is also briefly discussed. Four examples of branching processes, treated in Sec. V, drive home the message that multifractals contain less information than the spatial correlations. We discuss our results in more general terms in Sec. VI.

## II. MULTIPLICATIVE BRANCHING PROCESSES

Random multiplicative branching processes, sometimes also called weight-curdling models or simply cascade models, can be constructed for any number of branches per splitting. For simplicity, we concentrate on binary processes; generalizations are straightforward.

The branching rule governing a binary cascade is described as follows: a starting energy density  $\epsilon^{(0)} = 1$ , uniformly distributed on the unit interval, is split up into two daughter densities  $\epsilon_0^{(1)} = q_0 \epsilon^{(0)}$  and  $\epsilon_1^{(1)} = q_1 \epsilon^{(0)}$  living on adjacent subintervals of length  $2^{-1}$ . The random weights  $q_0$  and  $q_1$ , often also called multipliers or splitting parameters, are drawn from a joint probability density (or “splitting function”)  $p(q_0, q_1)$ . In the next generation, each of the two daughter energy densities is itself split up by the same branching rule into two granddaughters distributed uniformly over adjacent subintervals of length  $2^{-2}$ . Generally, energy densities of the  $j$ -th generation,  $\epsilon^{(j)}$ , are characterized by the binary index  $\kappa = (k_1 k_2 \dots k_j)$ , with each  $k$  taking on possible values 0 or 1. Successive repetition of this prescription for each  $\epsilon$  of the  $j$ -th generation yields, at the next branching,  $2^{j+1}$  energy densities  $\epsilon_{k_1 \dots k_j 0}^{(j+1)} = q_{k_1 \dots k_j 0}^{(j+1)} \epsilon_{k_1 \dots k_j}^{(j)}$  and  $\epsilon_{k_1 \dots k_j 1}^{(j+1)} = q_{k_1 \dots k_j 1}^{(j+1)} \epsilon_{k_1 \dots k_j}^{(j)}$ , with the multipliers drawn from a splitting function  $p(q_{k_1 \dots k_j 0}^{(j+1)}, q_{k_1 \dots k_j 1}^{(j+1)})$ , which usually (although not necessarily) is identical with  $p(q_0, q_1)$ . The process is completed after  $J$  cascade steps.

When the splitting function  $p(q_0, q_1)$  does not depend on the generation  $j$  or branch location  $k_1 \dots k_j$ , the above prescription gives rise to a selfsimilar multiplicative branching process, wholly characterized by the choice of splitting function  $p$ . We list four examples for later use. The choice

$$p(q_0, q_1) = \frac{1}{2} \left[ \delta(q_0 - (1 + \beta)) + \delta(q_0 - (1 - \beta)) \right] \delta(q_0 + q_1 - 2) \quad (1)$$

leads to the binomial multiplicative process, also known as the  $p$ -model (often the equivalent parameter  $p = (1 + \beta)/2$  is used) [8]. The delta function  $\delta(q_0 + q_1 - 2)$  ensures that this process conserves energy at every branching.

The splitting function for the non-energy conserving counterpart of the  $p$ -model, the  $\alpha$ -model [9], is given by

$$p(q_0, q_1) = \frac{1}{4} \prod_{k=0}^1 \left[ \delta(q_k - (1 + \beta)) + \delta(q_k - (1 - \beta)) \right] \quad (2)$$

For the energy dissipation process in fully developed turbulence, the parametrisation

$$\begin{aligned} p(q_0, q_1) &= \frac{1}{2} \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} \left( \frac{q_0}{2} \right)^{\beta-1} \left( \frac{q_1}{2} \right)^{\beta-1} \delta(q_0 + q_1 - 2) \\ &= \frac{1}{2^{2\beta-1}} \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} (q_0(2 - q_0))^{\beta-1} \delta(q_0 + q_1 - 2) \end{aligned} \quad (3)$$

with  $\beta = 3.2$  has been directly deduced from experiments [13]. The corresponding splitting function not conserving energy, is given by

$$p(q_0, q_1) = \left( \frac{1}{2^{2\beta-1}} \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} \right)^2 (q_0(2 - q_0))^{\beta-1} (q_1(2 - q_1))^{\beta-1} \quad (4)$$

These four splitting functions are shown in Fig. 1.

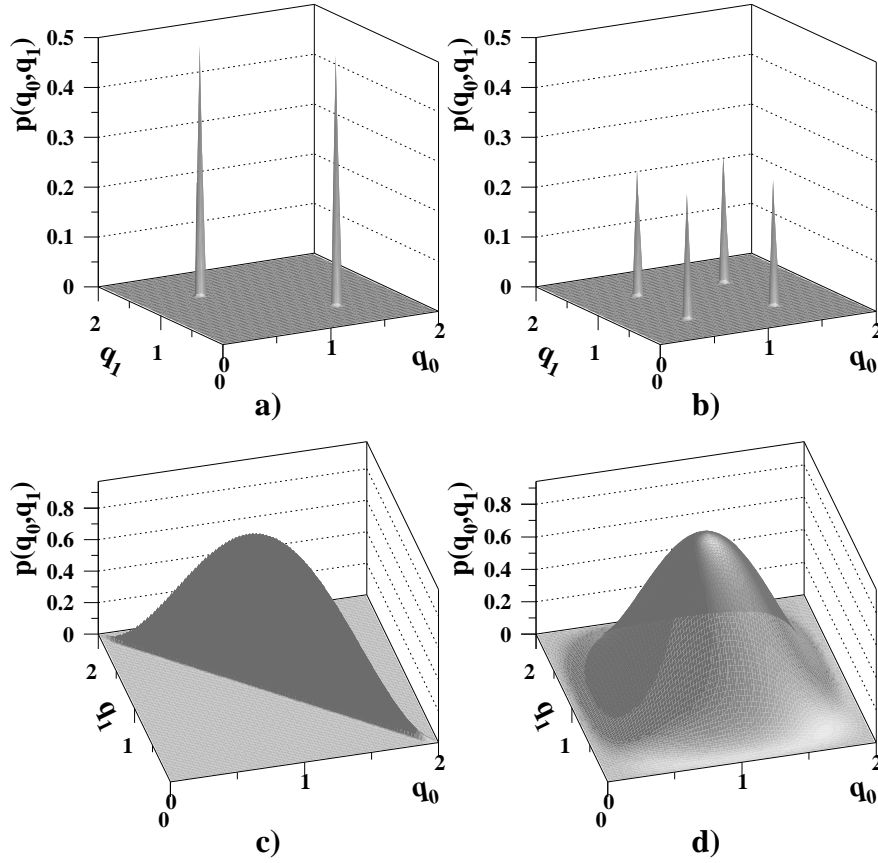


FIG. 1. Splitting functions  $p(q_0, q_1)$  of (a) the  $p$ -model with  $\beta = 0.4$ , (b) the  $\alpha$ -model with  $\beta = 0.4$ , (c) the energy-conserving SRST cascade model with  $\beta = 3.2$ , and (d) the energy-non-conserving SRST cascade model with  $\beta = 3.2$ .

### III. SPATIAL CORRELATIONS

#### A. Possible variables

For branching processes, the success in extracting useful information or presenting a clear picture depends strongly on the choice of variable. In Refs. [10,11,14], forward and backward evolution equations for a multivariate generating function were employed to calculate the correlations directly in terms of the energy densities,  $\langle \epsilon_{\kappa_1}^{(J)} \cdots \epsilon_{\kappa_n}^{(J)} \rangle$ . In the context of multifractals [1], on the other hand, another set of variables, the so-called singularity strengths  $\alpha_{\kappa}^{(J)}$ , are in use. The definition of these strengths,  $(2^{-J})\epsilon_{\kappa}^{(J)} = (2^{-J})^{\alpha_{\kappa}^{(J)}}$ , means that they are related to the energy density variables  $\epsilon_{\kappa}^{(J)}$  by

$$\alpha_{\kappa}^{(J)} = 1 - \frac{1}{J \ln 2} \ln \epsilon_{\kappa}^{(J)} \quad , \quad (5)$$

so that correlations  $\langle \alpha_{\kappa_1}^{(J)} \cdots \alpha_{\kappa_n}^{(J)} \rangle$  between the singularity strengths  $\alpha_{\kappa}^{(J)}$  are intimately related to corresponding correlations  $\langle (\ln \epsilon_{\kappa_1}^{(J)}) \cdots (\ln \epsilon_{\kappa_n}^{(J)}) \rangle$  between the logarithms of the energy densities  $\epsilon_{\kappa}^{(J)}$ .

We have shown previously [12] that correlations between the  $\ln \epsilon$  lead to a particularly transparent structure for multiplicative branching processes. Starting with correlations of the logarithms of energy densities, we show below how this transparent structure arises before translating these results into the language of multifractals.

## B. Correlations and generating functions

We consider the spatial correlations

$$\rho_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n} = \left\langle \ln \epsilon_{\mathbf{\kappa}_1}^{(J)} \cdots \ln \epsilon_{\mathbf{\kappa}_n}^{(J)} \right\rangle \quad (6)$$

of order  $n$  between the variables  $\ln \epsilon_{\mathbf{\kappa}}^{(J)}$ . The bracket  $\langle \dots \rangle$  indicates the averaging over all possible configurations. Moments  $\rho_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}$  are conveniently calculated from the multivariate generating function

$$Z[\lambda^{(J)}] = \left\langle \exp \left( \sum_{\substack{k_1, \dots, \\ k_J=0}}^1 \lambda_{k_1 \dots k_J}^{(J)} \ln \epsilon_{k_1 \dots k_J}^{(J)} \right) \right\rangle \quad (7)$$

by taking appropriate derivatives with respect to the conjugate variables  $\lambda_{\mathbf{\kappa}}^{(J)}$ :

$$\rho_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n} = \left. \frac{\partial^n Z[\lambda^{(J)}]}{\partial \lambda_{\mathbf{\kappa}_1}^{(J)} \cdots \partial \lambda_{\mathbf{\kappa}_n}^{(J)}} \right|_{\lambda=0} . \quad (8)$$

The corresponding multivariate cumulant generating function

$$K[\lambda^{(J)}] = \ln Z[\lambda^{(J)}] \quad (9)$$

yields the cumulant correlation densities in the same way via

$$C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n} = \left. \frac{\partial^n K[\lambda^{(J)}]}{\partial \lambda_{\mathbf{\kappa}_1}^{(J)} \cdots \partial \lambda_{\mathbf{\kappa}_n}^{(J)}} \right|_{\lambda^{(J)}=0} = \left\langle \ln \epsilon_{\mathbf{\kappa}_1}^{(J)} \cdots \ln \epsilon_{\mathbf{\kappa}_n}^{(J)} \right\rangle_c ; \quad (10)$$

here, we have introduced the index  $c$  (for “cumulant”) on the last bracket to distinguish (10) from the expression (6) for the ordinary correlation densities. The lowest four orders of cumulant correlation densities are

$$C_{\mathbf{\kappa}_1} = \langle \ln \epsilon_{\mathbf{\kappa}_1} \rangle_c = \langle \ln \epsilon_{\mathbf{\kappa}_1} \rangle , \quad (11)$$

$$\begin{aligned} C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2} &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \rangle_c \\ &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \rangle - \langle \ln \epsilon_{\mathbf{\kappa}_1} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_2} \rangle , \end{aligned} \quad (12)$$

$$\begin{aligned} C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2, \mathbf{\kappa}_3} &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \ln \epsilon_{\mathbf{\kappa}_3} \rangle_c \\ &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \ln \epsilon_{\mathbf{\kappa}_3} \rangle - \sum_{\{3\}} \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_3} \rangle \\ &\quad + 2 \langle \ln \epsilon_{\mathbf{\kappa}_1} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_2} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_3} \rangle , \end{aligned} \quad (13)$$

$$\begin{aligned} C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2, \mathbf{\kappa}_3, \mathbf{\kappa}_4} &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \ln \epsilon_{\mathbf{\kappa}_3} \ln \epsilon_{\mathbf{\kappa}_4} \rangle_c \\ &= \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \ln \epsilon_{\mathbf{\kappa}_3} \ln \epsilon_{\mathbf{\kappa}_4} \rangle - \sum_{\{4\}} \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \ln \epsilon_{\mathbf{\kappa}_3} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_4} \rangle \\ &\quad - \sum_{\{3\}} \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_3} \ln \epsilon_{\mathbf{\kappa}_4} \rangle + 2 \sum_{\{6\}} \langle \ln \epsilon_{\mathbf{\kappa}_1} \ln \epsilon_{\mathbf{\kappa}_2} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_3} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_4} \rangle \\ &\quad - 6 \langle \ln \epsilon_{\mathbf{\kappa}_1} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_2} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_3} \rangle \langle \ln \epsilon_{\mathbf{\kappa}_4} \rangle , \end{aligned} \quad (14)$$

with brackets  $\{ \dots \}$  indicating summation over permutations of indices.

### C. Construction of the cumulant generating function (first approach)

The derivation of the cumulant generating function depends crucially on a rearrangement of terms in the sum  $\sum_{k_1, \dots, k_J} \lambda_{k_1 \dots k_J}^{(J)} \ln \epsilon_{k_1 \dots k_J}^{(J)}$  entering the generating function in (7). The trick lies in recognising that the cascade prescription of Section II implies that the energy density in the  $J$ -th generation is the product of all the splitting parameters in its ancestry,

$$\epsilon_{\mathbf{k}}^{(J)} = \epsilon_{k_1 \dots k_J}^{(J)} = q_{k_1}^{(1)} q_{k_1 k_2}^{(2)} \cdots q_{k_1 \dots k_J}^{(J)} \quad , \quad (15)$$

so that its logarithm is additive in the multipliers' logarithms,

$$\ln \epsilon_{k_1 \dots k_J}^{(J)} = \sum_{j=1}^J \ln q_{k_1 \dots k_j}^{(j)} \quad . \quad (16)$$

Inserting this into the sum in the exponential of the generating function in (7) and defining

$$\lambda_{k_1 \dots k_j}^{(j)} \equiv \sum_{k_{j+1}, \dots, k_J} \lambda_{k_1 \dots k_j k_{j+1} \dots k_J}^{(J)} \quad , \quad (17)$$

we find, after rearrangement and judicious regrouping of terms, that

$$\begin{aligned} \sum_{k_1, \dots, k_J} \lambda_{k_1 \dots k_J}^{(J)} \ln \epsilon_{k_1 \dots k_J}^{(J)} &= \sum_{j=1}^J \sum_{k_1, \dots, k_j} \lambda_{k_1 \dots k_j}^{(j)} \ln q_{k_1 \dots k_j}^{(j)} \\ &= \sum_{j=1}^J \sum_{\substack{k_1, \dots, \\ k_{j-1}}} \left( \lambda_{k_1 \dots k_{j-1} 0}^{(j)} \ln q_{k_1 \dots k_{j-1} 0}^{(j)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} \ln q_{k_1 \dots k_{j-1} 1}^{(j)} \right) . \end{aligned} \quad (18)$$

This can now be utilized to rewrite the configuration average  $\langle \mathcal{O}(\epsilon^{(J)}) \rangle$  of any function  $\mathcal{O}$  in terms of the splitting parameters as follows. We start with the definition

$$\langle \mathcal{O}(\epsilon^{(J)}) \rangle = \int d(\ln \epsilon_{0 \dots 0}^{(J)}) \cdots d(\ln \epsilon_{1 \dots 1}^{(J)}) p(\ln \epsilon_{0 \dots 0}^{(J)}, \dots, \ln \epsilon_{1 \dots 1}^{(J)}) \mathcal{O}(\epsilon^{(J)}) \quad , \quad (19)$$

where  $p(\ln \epsilon_{0 \dots 0}^{(J)}, \dots, \ln \epsilon_{1 \dots 1}^{(J)})$  is the joint probability of finding a given logarithmic energy density  $\ln \epsilon_{0 \dots 0}^{(J)}$  in bin  $(0 \cdots 0)$ , a given  $\ln \epsilon_{0 \dots 1}^{(J)}$  in the next bin, etc. Since each energy density  $\epsilon_{\mathbf{k}}^{(J)}$  is completely determined by its  $J$  ancestral splitting parameters, the multivariate probability density  $p(\ln \epsilon_{0 \dots 0}^{(J)}, \dots, \ln \epsilon_{1 \dots 1}^{(J)})$  can be constructed from the splitting function  $p(q_0, q_1)$  by using (16):

$$\begin{aligned} p(\ln \epsilon_{0 \dots 0}^{(J)}, \dots, \ln \epsilon_{1 \dots 1}^{(J)}) &= \int \left[ \prod_{j=1}^J \prod_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 dq_{k_1 \dots k_{j-1} 0}^{(j)} dq_{k_1 \dots k_{j-1} 1}^{(j)} p(q_{k_1 \dots k_{j-1} 0}^{(j)}, q_{k_1 \dots k_{j-1} 1}^{(j)}) \right] \\ &\quad \times \left[ \prod_{\substack{k_1, \dots, \\ k_J=0}}^1 \delta \left( \ln \epsilon_{k_1 \dots k_J}^{(J)} - \sum_{j=1}^J \ln q_{k_1 \dots k_j}^{(j)} \right) \right] \quad . \end{aligned} \quad (20)$$

Inserting (18), (19) and (20) into Eq. (9), we find that the cumulant generating function becomes

$$\begin{aligned} K[\lambda_{0 \dots 0}^{(J)}, \dots, \lambda_{1 \dots 1}^{(J)}] &= \ln \left[ \int \left( \prod_{\substack{k_1, \dots, \\ k_J=0}}^1 d(\ln \epsilon_{k_1 \dots k_J}^{(J)}) \right) p(\ln \epsilon_{0 \dots 0}^{(J)}, \dots, \ln \epsilon_{1 \dots 1}^{(J)}) \right. \\ &\quad \times \exp \left( \sum_{\substack{k_1, \dots, \\ k_J=0}}^1 \lambda_{k_1 \dots k_J}^{(J)} \ln \epsilon_{k_1 \dots k_J}^{(J)} \right) \Big] \\ &= \sum_{j=1}^J \sum_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 Q[\lambda_{k_1 \dots k_{j-1} 0}^{(j)}, \lambda_{k_1 \dots k_{j-1} 1}^{(j)}] \quad , \end{aligned} \quad (21)$$

where

$$\begin{aligned} Q[\lambda_0, \lambda_1] &= \ln \left[ \int dq_0 dq_1 p(q_0, q_1) \exp(\lambda_0 \ln q_0 + \lambda_1 \ln q_1) \right] \\ &= \ln \left\langle \exp(\lambda_0 \ln q_0 + \lambda_1 \ln q_1) \right\rangle \end{aligned} \quad (22)$$

Eqs. (21)–(22) represent the long-sought analytic expression for multiplicative cascades: they show that the cumulant generating function of the entire cascade can be written as the sum of all branching generating functions  $Q$ , one for every branching. In most cases, the two-fold integrals (22) can be solved analytically or parametrically to yield a complete analytic solution for  $K$  and thereby for all cumulants.

The scope and limitations of the solution (21) are as follows. Clearly, the generating function in terms of  $\ln \epsilon$  is applicable to any functional form of the splitting function or b.g.f. It does not depend on the number of branches either: trivariate or even higher-variate splitting functions can be implemented. Due to the additive nature of the b.g.f.'s, the splitting functions can differ from generation to generation and even from branch to branch. The only (and important) precondition for the applicability of Eq. (21) is that the splitting variables of every branching must be independent of those of the other branchings and generations.

#### D. Construction of the cumulant generating function (second approach)

The cumulant generating function can also be constructed from a forward evolution equation. In Ref. [14], the forward evolution equation for the generating function was derived in terms of energy densities  $\epsilon_{k_1 \dots k_j}^{(j)}$ . In terms of the  $\ln \epsilon_{k_1 \dots k_j}^{(j)}$ , the cumulant generating function after  $j$  cascade steps,  $K^{(j)}[\lambda^{(j)}]$ , can be expressed in terms of  $K^{(j-1)}[\lambda^{(j-1)}]$  as follows. At each of the  $2^{j-1}$  independent branchings, Eq. (16) relates the daughter energy densities to their parent by

$$\ln \epsilon_{k_1 \dots k_j}^{(j)} = \ln \epsilon_{k_1 \dots k_{j-1} 0}^{(j-1)} + \ln q_{k_1 \dots k_j}^{(j)} \quad (k_j = 0, 1). \quad (23)$$

In analogy to (18),

$$\begin{aligned} &\exp \left( \sum_{\substack{k_1, \dots, \\ k_j=0}}^1 \lambda_{k_1 \dots k_j}^{(j)} \ln \epsilon_{k_1 \dots k_j}^{(j)} \right) = \\ &= \prod_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 \exp \left( \lambda_{k_1 \dots k_{j-1} 0}^{(j)} \ln \epsilon_{k_1 \dots k_{j-1} 0}^{(j-1)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} \ln \epsilon_{k_1 \dots k_{j-1} 1}^{(j-1)} \right) \\ &= \prod_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 \exp \left[ \left( \lambda_{k_1 \dots k_{j-1} 0}^{(j)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} \right) \ln \epsilon_{k_1 \dots k_{j-1}}^{(j-1)} \right. \\ &\quad \left. + \lambda_{k_1 \dots k_{j-1} 0}^{(j)} \ln q_{k_1 \dots k_{j-1} 0}^{(j)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} \ln q_{k_1 \dots k_{j-1} 1}^{(j)} \right] \end{aligned} \quad (24)$$

and assigning, as in (17),

$$\lambda_{k_1 \dots k_{j-1} 0}^{(j)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} = \lambda_{k_1 \dots k_{j-1}}^{(j-1)}, \quad (25)$$

we arrive at

$$\begin{aligned} K^{(j)}[\lambda^{(j)}] &= K^{(j-1)}[\lambda^{(j-1)}] \\ &+ \sum_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 \ln \left[ \int dq_{k_1 \dots k_{j-1} 0}^{(j)} dq_{k_1 \dots k_{j-1} 1}^{(j)} p(q_{k_1 \dots k_{j-1} 0}^{(j)}, q_{k_1 \dots k_{j-1} 1}^{(j)}) \right. \\ &\quad \left. \times \exp \left( \lambda_{k_1 \dots k_{j-1} 0}^{(j)} \ln q_{k_1 \dots k_{j-1} 0}^{(j)} + \lambda_{k_1 \dots k_{j-1} 1}^{(j)} \ln q_{k_1 \dots k_{j-1} 1}^{(j)} \right) \right] \\ &= K^{(j-1)}[\lambda^{(j-1)}] + \sum_{\substack{k_1, \dots, \\ k_{j-1}=0}}^1 Q[\lambda_{k_1 \dots k_{j-1} 0}^{(j)}, \lambda_{k_1 \dots k_{j-1} 1}^{(j)}] \quad . \end{aligned} \quad (26)$$

This forward evolution equation can be iterated from  $j = J$  down to  $j = 0$ , where we arrive at  $K^{(j=0)}[\lambda^{(0)}] = \ln(Z^{(0)}[\lambda^{(0)}]) = 0$ ; the outcome of this iteration is identical to the solution (21). Similarly, a backward evolution equation for the cumulant generating functions can also be used to achieve identical results; we leave this as an exercise to the reader.

### E. Cumulant correlation densities

Having found an analytic expression of the multivariate generating function  $K[\lambda^{(J)}]$ , it is now straightforward to calculate the cumulant correlation densities (11)–(14) between the  $\ln \epsilon_{\mathbf{\kappa}}^{(J)}$ -variables via Eq. (10). In order to give compact expressions, we define an ultrametric distance between bins. Assume the two bins are together for the first  $j$  steps of the binary cascade before splitting at the  $(j+1)$ -th generation,  $\mathbf{\kappa}_1 = (k_1 \cdots k_j k_{j+1} \cdots k_J)$  and  $\mathbf{\kappa}_2 = (k_1 \cdots k_j k'_{j+1} \cdots k'_J)$ , where  $k_i = k'_i$  for all  $1 \leq i \leq j$  and  $k_{j+1} \neq k'_{j+1}$ . Then the ultrametric distance between them is

$$d_2 = \text{dist}(\mathbf{\kappa}_1, \mathbf{\kappa}_2) = J - j \quad . \quad (27)$$

For  $n$  bins, the generalised ultrametric distance is

$$d_n = \max_{1 \leq i < j \leq n} \text{dist}(\mathbf{\kappa}_i, \mathbf{\kappa}_j) \quad . \quad (28)$$

In the following, we assume that the same functional form for the b.g.f. is used at all branchings. We also assume the splitting function  $p(q_0, q_1)$  to be symmetric under exchange of the splitting parameters,  $p(q_0, q_1) = p(q_1, q_0)$ ; the four splitting functions given in Eqs. (1)–(4) fall into this category. In consequence, the branching generating function  $Q[\lambda_0, \lambda_1] = Q[\lambda_1, \lambda_0]$  is also symmetric, so that also

$$\left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n_2}} \right|_{\lambda=0} = \left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_2} \partial \lambda_1^{n_1}} \right|_{\lambda=0} \quad . \quad (29)$$

This relation results in compact expressions for the cumulant correlation densities  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}$ . Inserting (21) into Eq. (10) and taking into account (17), the cumulant correlation density of first order is found to be

$$C_{\mathbf{\kappa}_1} = J \left( \left. \frac{\partial Q[\lambda_0, \lambda_1]}{\partial \lambda_0} \right|_{\lambda=0} \right) \quad . \quad (30)$$

For second order we get, with (27),

$$C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2} = (J - d_2) \left( \left. \frac{\partial^2 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^2} \right|_{\lambda=0} \right) + (1 - \delta_{d_2,0}) \left( \left. \frac{\partial^2 Q[\lambda_0, \lambda_1]}{\partial \lambda_0 \partial \lambda_1} \right|_{\lambda=0} \right) \quad , \quad (31)$$

while with the generalisation (28), the third order cumulant density becomes

$$C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2, \mathbf{\kappa}_3} = (J - d_3) \left( \left. \frac{\partial^3 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^3} \right|_{\lambda=0} \right) + (1 - \delta_{d_3,0}) \left( \left. \frac{\partial^3 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^2 \partial \lambda_1} \right|_{\lambda=0} \right) \quad . \quad (32)$$

In the case of fourth order, two cases have to be distinguished, depending on whether the first splitting of the four bins  $\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_4$  goes into three and one, say  $(\mathbf{\kappa}_1 \mathbf{\kappa}_2 \mathbf{\kappa}_3 | \mathbf{\kappa}_4)$  with  $d_3(\mathbf{\kappa}_1, \mathbf{\kappa}_2, \mathbf{\kappa}_3) < d_4$  and  $n_1 = 3, n_2 = 1$ , or into two and two, say  $(\mathbf{\kappa}_1 \mathbf{\kappa}_2 | \mathbf{\kappa}_3 \mathbf{\kappa}_4)$  with  $d_2(\mathbf{\kappa}_1, \mathbf{\kappa}_2) < d_4$ ,  $d_2(\mathbf{\kappa}_3, \mathbf{\kappa}_4) < d_4$  and  $n_1 = 2, n_2 = 2$ :

$$C_{\mathbf{\kappa}_1, \mathbf{\kappa}_2, \mathbf{\kappa}_3, \mathbf{\kappa}_4} = (J - d_4) \left( \left. \frac{\partial^4 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^4} \right|_{\lambda=0} \right) + (1 - \delta_{d_4,0}) \left( \left. \frac{\partial^4 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^2 \partial \lambda_1^2} \right|_{\lambda=0} \right) \quad . \quad (33)$$

Even under the stated assumptions, the expressions (30)–(33) are still very general for selfsimilar binary multiplicative cascade models. The one and only input is the splitting function  $p(q_0, q_1)$  determining the branching generating function  $Q[\lambda_0, \lambda_1]$ . The cumulant correlation densities for the splitting functions (1)–(4) will be discussed in more detail in Sec. V.

It is clear from Eqs. (30)–(33) that the derivatives of the branching generating function completely fix the spatial cumulant correlation densities. Using the definition (22) these derivatives can be expressed in terms of generic branching moments:

$$\left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n_2}} \right|_{\lambda=0} = \langle (\ln q_0)^{n_1} (\ln q_1)^{n_2} \rangle_c . \quad (34)$$

In the lowest three orders, they read explicitly:

$$\begin{aligned} \langle \ln q_0 \rangle_c &= \langle \ln q_0 \rangle , \\ \langle (\ln q_0)^2 \rangle_c &= \langle (\ln q_0)^2 \rangle - \langle \ln q_0 \rangle^2 , \\ \langle (\ln q_0)(\ln q_1) \rangle_c &= \langle (\ln q_0)(\ln q_1) \rangle - \langle \ln q_0 \rangle \langle \ln q_1 \rangle , \\ \langle (\ln q_0)^3 \rangle_c &= \langle (\ln q_0)^3 \rangle - 3 \langle (\ln q_0)^2 \rangle \langle \ln q_0 \rangle + 2 \langle \ln q_0 \rangle^3 , \\ \langle (\ln q_0)^2 (\ln q_1) \rangle_c &= \langle (\ln q_0)^2 (\ln q_1) \rangle - 2 \langle (\ln q_0)(\ln q_1) \rangle \langle \ln q_0 \rangle - \langle (\ln q_0)^2 \rangle \langle \ln q_1 \rangle \\ &\quad + 2 \langle \ln q_0 \rangle^2 \langle \ln q_1 \rangle . \end{aligned} \quad (35)$$

### F. Link to singularity strength variables

We have defined the cumulant correlation densities  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n} = \langle \ln \epsilon_{\mathbf{\kappa}_1}^{(J)} \cdots \ln \epsilon_{\mathbf{\kappa}_n}^{(J)} \rangle_c$  in terms of the  $\ln \epsilon_{\mathbf{\kappa}}^{(J)}$ -variables. With the help of (5), they can be transformed into cumulant correlation densities  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}^\alpha = \langle \alpha_{\mathbf{\kappa}_1}^{(J)} \cdots \alpha_{\mathbf{\kappa}_n}^{(J)} \rangle_c$  between the singularity strength variables  $\alpha_{\mathbf{\kappa}}^{(J)}$ . We find for the corresponding generating function

$$\begin{aligned} K^\alpha[\lambda^{(J)}] &= \ln \left\langle \exp \left( \sum_{\substack{k_1, \dots, \\ k_J=0}}^1 \lambda_{k_1 \dots k_J}^{(J)} \alpha_{k_1 \dots k_J}^{(J)} \right) \right\rangle \\ &= \ln \left\langle \exp \left( \sum_{\substack{k_1, \dots, \\ k_J=0}}^1 \lambda_{k_1 \dots k_J}^{(J)} \left( 1 - \frac{1}{J \ln 2} \ln \epsilon_{k_1 \dots k_J}^{(J)} \right) \right) \right\rangle \\ &= \sum_{\substack{k_1, \dots, \\ k_J=0}}^1 \lambda_{k_1 \dots k_J}^{(J)} + K \left[ \frac{(-1)}{J \ln 2} \lambda^{(J)} \right] . \end{aligned} \quad (36)$$

Taking derivatives, the relations between the cumulant densities are found to be, for first order,

$$C_{\mathbf{\kappa}_1}^\alpha = \langle \alpha_{\mathbf{\kappa}_1}^{(J)} \rangle_c = 1 - \frac{1}{J \ln 2} \langle \ln \epsilon_{\mathbf{\kappa}_1}^{(J)} \rangle_c = 1 - \frac{1}{J \ln 2} C_{\mathbf{\kappa}_1} , \quad (37)$$

and, for  $n \geq 2$ ,

$$C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}^\alpha = \langle \alpha_{\mathbf{\kappa}_1}^{(J)} \cdots \alpha_{\mathbf{\kappa}_n}^{(J)} \rangle_c = \left( \frac{-1}{J \ln 2} \right)^n \langle \ln \epsilon_{\mathbf{\kappa}_1}^{(J)} \cdots \ln \epsilon_{\mathbf{\kappa}_n}^{(J)} \rangle_c = \left( \frac{-1}{J \ln 2} \right)^n C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n} , \quad (38)$$

in other words, with the exception of first order, the  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}^\alpha$  are directly proportional to  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}$ .

## IV. SPATIAL CORRELATIONS VS. MULTIFRACTALS

### A. Multifractals

One way to approach multifractals is to count how often specific values  $\alpha$  of the singularity strengths occur; this leads to the  $f(\alpha)$ -spectrum [1]. The latter is related by a Legendre transformation to a sequence of mass exponents  $\tau(\nu)$  which are defined by the moment scaling behavior of bin energies  $E_{k_1 \dots k_j}^{(j)}$ :

$$M_\nu(j) = \left\langle \sum_{\substack{k_1, \dots, \\ k_j=0}}^1 \left( E_{k_1 \dots k_j}^{(j)} \right)^\nu \right\rangle = \left\langle \sum_{\substack{k_1, \dots, \\ k_j=0}}^1 \left( \frac{\epsilon_{k_1 \dots k_j}^{(j)}}{2^j} \right)^\nu \right\rangle = (2^{-j})^{-\tau(\nu)} . \quad (39)$$



In order to avoid confusion we have written  $\tau(\nu)$  instead of the more familiar notation  $\tau(q)$ . Since the bin energies are related to the singularity strengths by  $E_{\mathbf{k}}^{(j)} = (2^{-j})^{\alpha_{\mathbf{k}}^{(j)}}$ , these moments are to be understood as scale-dependent measures for the one-point statistics of the singularity strengths.

We first derive the expression for the exponents  $\tau(\nu)$  for the special case of a symmetric splitting function  $p(q_0, q_1) = p(q_1, q_0)$  of a binary multiplicative cascade process which conserves energy. (Note that for energy non-conserving splitting functions, as for example those given in (2) and (4), the multifractal exponents cannot be defined unambiguously; see Ref. [11].) Then we can use (15) and the statistical independence of splitting parameters to deduce

$$\left\langle \left( \epsilon_{k_1 \dots k_j}^{(j)} \right)^\nu \right\rangle = \langle q_{k_1}^\nu \rangle \langle q_{k_1 k_2}^\nu \rangle \cdots \langle q_{k_1 \dots k_j}^\nu \rangle = \langle q_0^\nu \rangle^j \quad . \quad (40)$$

Insertion into Eq. (39) yields:

$$\tau(\nu) = \frac{1}{\ln 2} \ln \left( \frac{\langle q_0^\nu \rangle}{2^{\nu-1}} \right) = -(\nu - 1) + \frac{\ln \langle q_0^\nu \rangle}{\ln 2} \quad . \quad (41)$$

For a  $q_0/q_1$ -asymmetric, energy-conserving splitting function we have to replace the sum appearing in (39) by

$$\sum_{\substack{k_1, \dots, \\ k_j=0}}^1 \left\langle \left( E_{k_1 \dots k_j}^{(j)} \right)^\nu \right\rangle = \frac{1}{2^{\nu j}} [\langle q_0^\nu \rangle + \langle q_1^\nu \rangle]^j \quad , \quad (42)$$

which gives rise to

$$\tau(\nu) = \frac{1}{\ln 2} \ln \left( \frac{\langle q_0^\nu \rangle + \langle q_1^\nu \rangle}{2^\nu} \right) = -\nu + \frac{\ln (\langle q_0^\nu \rangle + \langle q_1^\nu \rangle)}{\ln 2} \quad . \quad (43)$$

## B. Relationship between $Q[\lambda_0, \lambda_1]$ and $\tau(\nu)$

The close relationship between the branching generating function  $Q[\lambda_0, \lambda_1]$  and the multifractal exponents  $\tau(\nu)$  is revealed by simple inspection of (22) and (43). We find that

$$\tau(\nu) = -\nu + \frac{1}{\ln 2} \ln \left[ \exp(Q[\lambda_0 = \nu, \lambda_1 = 0]) + \exp(Q[\lambda_0 = 0, \lambda_1 = \nu]) \right] \quad . \quad (44)$$

This relationship<sup>1</sup> simplifies further once we consider an energy-conserving,  $q_0/q_1$  symmetric splitting function. Through (41) we arrive at

$$\tau(\nu) = -(\nu - 1) + \frac{1}{\ln 2} Q[\lambda_0 = \nu, \lambda_1 = 0] \quad . \quad (45)$$

The two relations (44) and (45) express  $\tau(\nu)$  in terms of  $Q[\lambda_0, \lambda_1]$ . It appears that  $\tau$  is more limited than  $Q$  in that the latter is defined over the full  $\lambda_0$ - $\lambda_1$  plane while  $\tau(\nu)$  is restricted to the axes  $\lambda_0 = 0$  or  $\lambda_1 = 0$ .

Let us consider the point in more detail. In particular, we will demonstrate that different energy-conserving splitting functions may lead to identical multifractal exponents while their branching generating functions do differ.

Given an energy-conserving,  $q_0/q_1$ -symmetric splitting function  $p(q_0, q_1) = p(q_1, q_0) = p(q_0)\delta(q_0 + q_1 - 2)$ , we construct a related splitting function by breaking the  $q_0/q_1$ -symmetry:

$$\tilde{p}(q_0, q_1) = \begin{cases} 2p(q_0, q_1) & (0 \leq q_0 \leq 1 \leq q_1 \leq 2) \\ 0 & (0 \leq q_1 \leq 1 \leq q_0 \leq 2) \end{cases} \quad . \quad (46)$$

The splitting function  $\tilde{p}(q_0, q_1)$  is almost identical to  $p(q_0, q_1)$ , their only difference being that the smaller splitting parameter  $q_0$  is now always drawn for the branching into the left subinterval whereas the the larger splitting parameter

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<sup>1</sup> The univariate version of this relation has already been discussed by Novikov [15] in connection with the statistics of generalised multipliers, the so-called breakdown coefficients.

$q_1 = 2 - q_0$  goes into the right subinterval. The one-point statistic is not affected by this modification of the splitting function, since it does not care whether the larger splitting parameter goes to the left or right subinterval. Hence the multifractal exponents are not changed either. With (43) we get

$$\begin{aligned}
\tilde{\tau}(\nu) &= -\nu + \frac{1}{\ln 2} \ln \left( \int_0^2 dq_0 dq_1 \tilde{p}(q_0, q_1) (q_0^\nu + q_1^\nu) \right) \\
&= -\nu + \frac{1}{\ln 2} \ln \left( 2 \int_0^1 dq_0 p(q_0) (q_0^\nu + (2 - q_0)^\nu) \right) \\
&= -(\nu - 1) + \frac{1}{\ln 2} \ln \left( \int_0^2 dq_0 p(q_0) q_0^\nu \right) \\
&= \tau(\nu) \quad .
\end{aligned} \tag{47}$$

We are hence forced to conclude that a multifractal analysis does not suffice completely to characterize the dynamics, i.e. the splitting function, of the underlying multiplicative cascade process. This statement holds even more strongly once we also give up energy conservation in the splitting function, with the consequence that properly defined backward moments show deviations from perfect multifractal scaling. In order to attain the full information, there is no way to get around spatial correlations in general, and the cumulant correlation densities  $C_{\kappa_1, \dots, \kappa_n}$  in particular, to extract the branching moments  $\langle (\ln q_0)^{n-m} (\ln q_1)^m \rangle_c$  and hence the branching generating function

$$Q[\lambda_0, \lambda_1] = \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1! n_2!} \langle (\ln q_0)^{n_1} (\ln q_1)^{n_2} \rangle_c \lambda_0^{n_1} \lambda_1^{n_2} \quad . \tag{48}$$

Using Eq. (22) the branching generating function can then be uniquely inverted into the splitting function via a two-dimensional inverse Laplace transformation,

$$\int_0^\infty dx dy p(2e^{-x}, 2e^{-y}) e^{-(\lambda_0+1)x - (\lambda_1+1)y} = e^{Q[\lambda_0, \lambda_1] - (\lambda_0 + \lambda_1 + 2) \ln 2} \quad . \tag{49}$$

As a consequence we view  $Q[\lambda_0, \lambda_1]$  as the natural and complete generalisation of the multifractal mass exponents  $\tau(\nu)$  for random multiplicative binary cascade processes.

## V. EXAMPLES

### A. Multiplicative binomial process: the $p$ -model

The  $p$ -model splitting function given in Eq. (1), when inserted into Eq. (22), determines the branching generating function  $Q[\lambda_0, \lambda_1]$ :

$$\begin{aligned}
Q[\lambda_0, \lambda_1] &= \ln \left( \frac{1}{2} \{ \exp [\lambda_0 \ln(1 + \beta) + \lambda_1 \ln(1 - \beta)] \right. \\
&\quad \left. + \exp [\lambda_0 \ln(1 - \beta) + \lambda_1 \ln(1 + \beta)] \} \right) \quad .
\end{aligned} \tag{50}$$

We introduce the transformation

$$\lambda_0 = \frac{1}{2} (\lambda_+ + \lambda_-) \quad , \quad \lambda_1 = \frac{1}{2} (\lambda_+ - \lambda_-) \quad , \tag{51}$$

which leads to

$$Q[\lambda_0, \lambda_1] = \frac{\ln(1 - \beta^2)}{2} \lambda_+ + \ln \left( \cosh \left( \frac{1}{2} \lambda_- \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right) \right) \quad . \tag{52}$$

Fig. 2(a) illustrates this branching generating function.

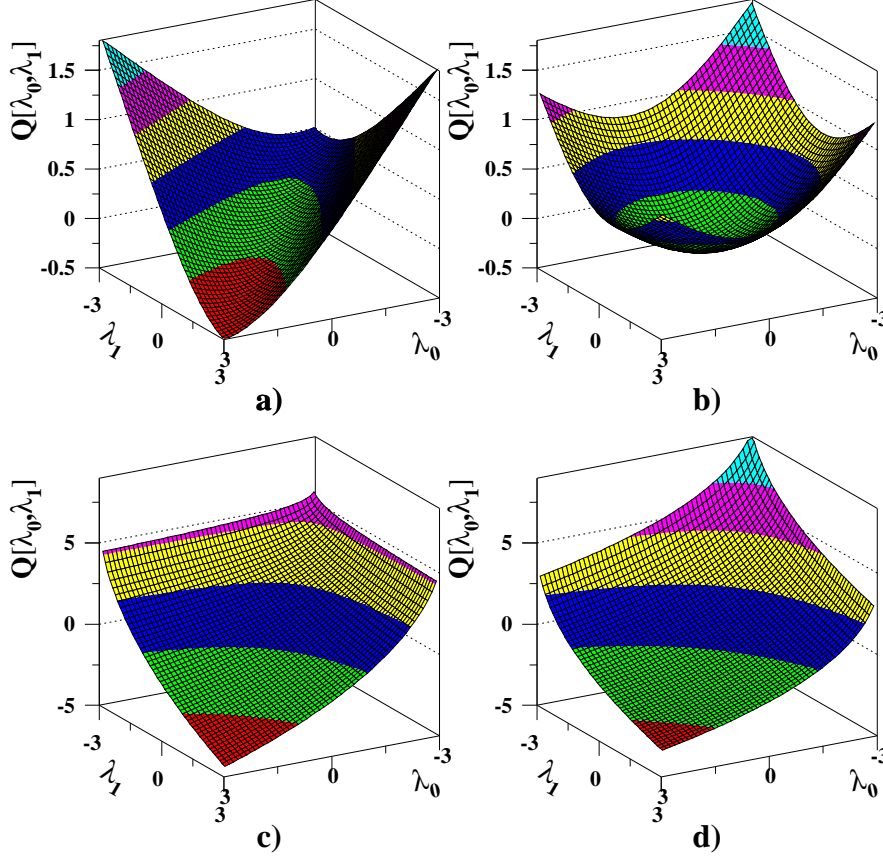


FIG. 2. Branching generating function  $Q[\lambda_0, \lambda_1]$ , shown in the range  $-3 \leq \lambda_0, \lambda_1 \leq 3$ , for (a) the  $p$ -model with  $\beta = 0.4$ , (b) the  $\alpha$ -model with  $\beta = 0.4$ , (c) the energy-conserving SRST cascade model with  $\beta = 3.2$ , and (d) the energy-non-conserving SRST cascade model with  $\beta = 3.2$ .

Now we calculate the derivatives of the branching generating function with respect to the conjugate variables  $\lambda_0$  and  $\lambda_1$ , which are needed for the cumulant densities  $C_{\kappa_1, \dots, \kappa_n}$ . For the first derivative we find:

$$\begin{aligned}
 \left. \frac{\partial Q[\lambda_0, \lambda_1]}{\partial \lambda_0} \right|_{\lambda=0} &= \left. \frac{\partial Q[\lambda_0, \lambda_1]}{\partial \lambda_1} \right|_{\lambda=0} \\
 &= \frac{\ln(1 - \beta^2)}{2} + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \tanh \left( \frac{1}{2} \lambda_- \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right) \Big|_{\lambda=0} \\
 &= \frac{1}{2} \ln(1 - \beta^2) \quad .
 \end{aligned} \tag{53}$$

For an arbitrary derivative of  $Q[\lambda_0, \lambda_1]$  of order  $n_1 + n_2 > 1$ , we get the following relationship:

$$\begin{aligned}
 \left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n_2}} \right|_{\lambda=0} &= \left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_-^{n_1+n_2}} \right|_{\lambda=0} \left( \frac{\partial \lambda_-}{\partial \lambda_0} \right)^{n_1} \left( \frac{\partial \lambda_-}{\partial \lambda_1} \right)^{n_2} \\
 &= (-1)^{n_2} \left. \frac{\partial^{n_1+n_2} Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1+n_2}} \right|_{\lambda=0} \quad .
 \end{aligned} \tag{54}$$

Except for an alternating sign, the two-point branching moments  $\langle (\ln q_0)^{n_1} (\ln q_1)^{n_2} \rangle_c$  are identical to the one-point branching moments  $\langle (\ln q_0)^{n_1+n_2} \rangle_c$ ; this only holds for the  $p$ -model and not for the other three models associated with the splitting functions (2)–(4). As a consequence of (54), we only need to calculate derivatives of  $Q[\lambda_0, \lambda_1]$  with respect to  $\lambda_0$ . We use the intermediate step of (53) and write for  $n > 1$ :

$$\left. \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \right|_{\lambda=0} = \frac{1}{2} \ln \left( \frac{1+\beta}{1-\beta} \right) \left( \frac{\partial^{n-1}}{\partial \lambda_0^{n-1}} \left\{ \tanh \left[ \frac{1}{2} \lambda_0 \ln \left( \frac{1+\beta}{1-\beta} \right) \right] \right\} \right) \Big|_{\lambda_0=0}. \quad (55)$$

Since  $\tanh x$  is an odd function in  $x$ , i.e.

$$\begin{aligned} \tanh x &= x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 \mp \dots \\ &= \sum_{m=1}^{\infty} \frac{2^{2m}(2^{2m}-1)B_{2m}}{(2m)!} x^{2m-1} \end{aligned} \quad (56)$$

with the Bernoulli numbers  $B_{2m}$ , all odd derivatives of the branching generating function  $Q[\lambda_0, \lambda_1]$  with respect to  $\lambda_0$  vanish:

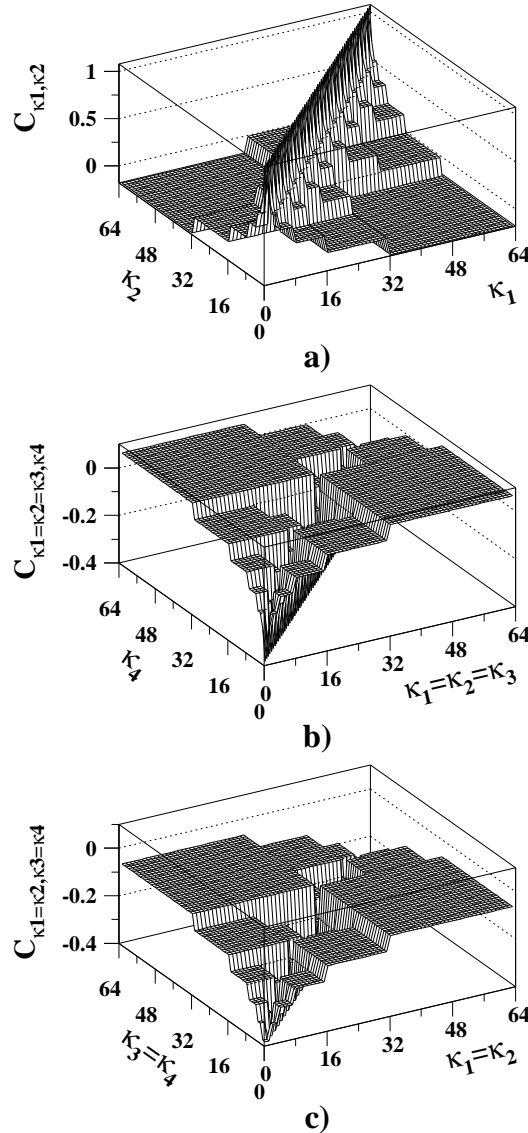


FIG. 3. Spatial cumulant densities for the  $p$ -model with  $\beta = 0.4$  and  $J = 6$ : (a) second order  $C_{\kappa_1, \kappa_2}$ , (b) fourth order with three equal indices,  $C_{\kappa_1=\kappa_2=\kappa_3, \kappa_4}$ , and (c) fourth order with two pairs of equal indices,  $C_{\kappa_1=\kappa_2, \kappa_3=\kappa_4}$ . The indicated bin labels are related to the binary indices  $\kappa = (k_1 \dots k_J)$  by  $k = 1 + \sum_{j=1}^J k_j 2^{J-j}$ .

$$\left. \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \right|_{\lambda=0} = 0 \quad (n = 3, 5, 7, \dots) \quad . \quad (57)$$

The same conclusion could have also been obtained directly from (54): for odd  $n \geq 3$  we have  $\partial^n Q[\lambda_0, \lambda_1]/\partial \lambda_0^n|_{\lambda=0} = -\partial^n Q[\lambda_0, \lambda_1]/\partial \lambda_1^n|_{\lambda=0}$ , but from  $q_0/q_1$ -symmetry considerations of the splitting function we expect  $\langle (\ln q_0)^n \rangle_c = \langle (\ln q_1)^n \rangle_c$ ; hence,  $\partial^n Q[\lambda_0, \lambda_1]/\partial \lambda_0^n|_{\lambda=0} = \langle (\ln q_0)^n \rangle_c = 0$ . An immediate consequence of this result is that, for example, the spatial cumulant densities of third order vanish,  $C_{\kappa_1, \kappa_2, \kappa_3} = 0$ ; see Eq. (32).

For even derivatives of  $Q[\lambda_0, \lambda_1]$  with respect to  $\lambda_0$  we find from (55) and (56) that

$$\left. \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \right|_{\lambda=0} = \frac{(2^n - 1)B_n}{n} \left( \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right)^n \quad (n = 2, 4, 6, \dots) \quad . \quad (58)$$

The results (53), (54), (57) and (58) can now be inserted into Eqs. (30)–(33) to determine the cumulant correlation densities  $C_{\kappa_1, \dots, \kappa_n} = \langle (\ln \epsilon_{\kappa_1}) \cdots (\ln \epsilon_{\kappa_n}) \rangle_c$ . Fig. 3 illustrates the results for the two-point statistics:  $C_{\kappa_1, \kappa_2}$  of second order, and  $C_{\kappa_1 = \kappa_2 = \kappa_3, \kappa_4}$  and  $C_{\kappa_1 = \kappa_2, \kappa_3 = \kappa_4}$  of fourth order; note again that the third order  $C_{\kappa_1, \kappa_2, \kappa_3}$ , vanishes. Figs. 4(a) and 4(b) show the second-order and fourth order cumulants as a function of the ultrametric distances respectively.

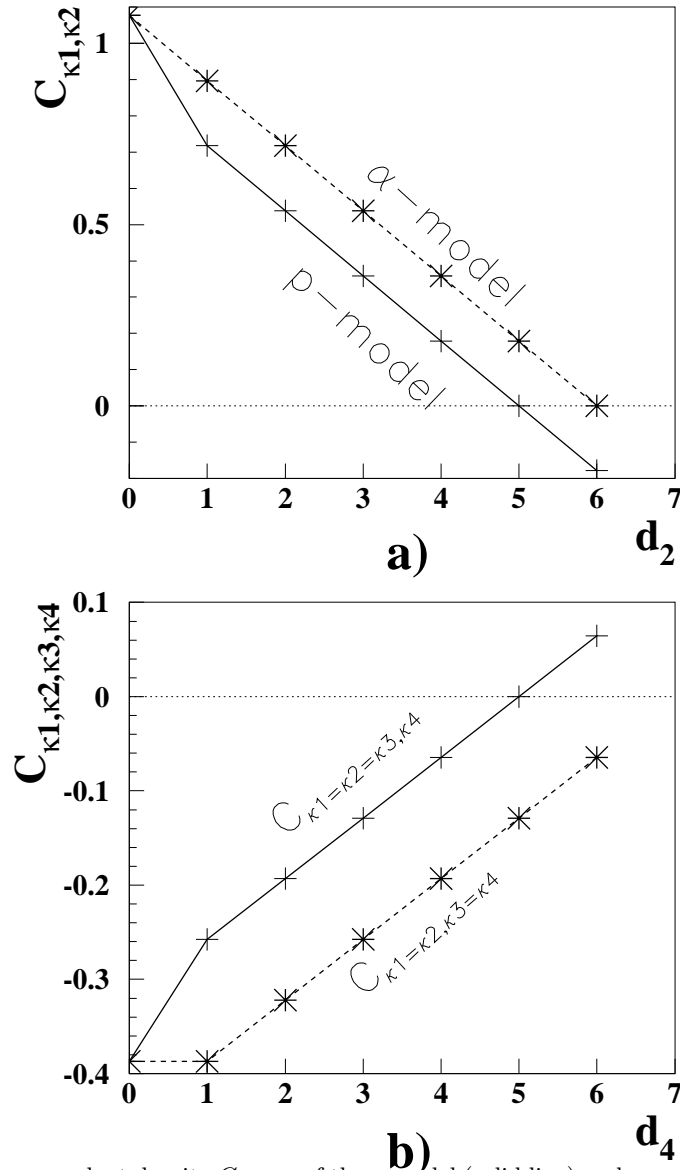


FIG. 4. (a) Second-order cumulant density  $C_{\kappa_1, \kappa_2}$  of the  $p$ -model (solid line) and  $\alpha$ -model (dashed line) as a function of the ultrametric distance  $d_2 = \text{dist}(\kappa_1, \kappa_2)$ . Parameters values  $\beta = 0.4$  and  $J = 6$  were used. (b) Two projections of fourth-order cumulant densities for the  $p$ -model as a function of the ultrametric distance  $d_4$ .

Since the  $p$ -model splitting function is energy-conserving and  $q_0/q_1$ -symmetric we can use relation (45) to extract the multifractal exponents  $\tau(\nu)$  from the branching generating function  $Q[\lambda_0, \lambda_1]$ . This leads to the well-known result [1,8]

$$\tau(\nu) = \frac{1}{\ln 2} \ln \left[ \left( \frac{1+\beta}{2} \right)^\nu + \left( \frac{1-\beta}{2} \right)^\nu \right] . \quad (59)$$

It is illustrated in Fig. 5. Via

$$\begin{aligned} \left. \frac{\partial^n \tau(\nu)}{\partial \nu^n} \right|_{\nu=0} &= -\delta_{n,1} + \frac{1}{\ln 2} \left. \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \right|_{\lambda=0} \\ &= -\delta_{n,1} + \frac{1}{\ln 2} \langle (\ln q_0)^n \rangle_c \end{aligned} \quad (60)$$

the derivatives of  $\tau(\nu)$  with respect to  $\nu$  are linked to the derivatives (53), (57) and (58) of the  $p$ -model branching generating function (52).

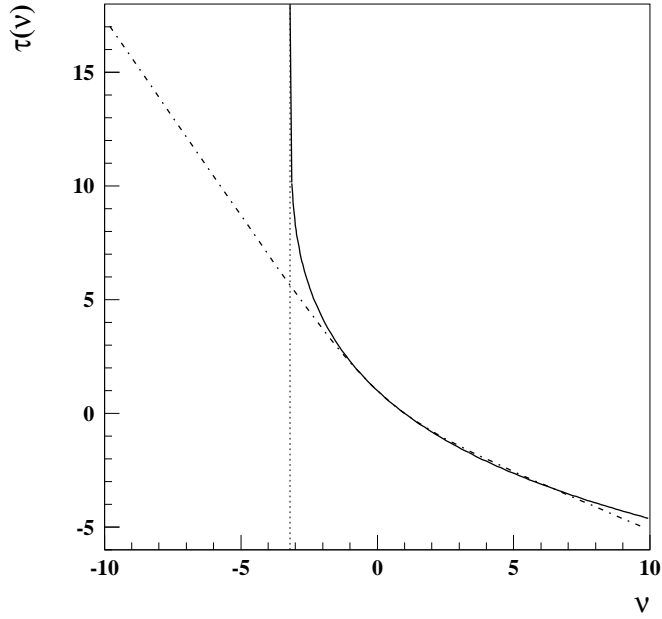


FIG. 5. Multifractal exponents  $\tau(\nu)$  of the  $p$ -model (dash-dotted line) with  $\beta = 0.4$  and of the SRST cascade model (full line) with  $\beta = 3.2$ .

## B. $\alpha$ -model

The symmetric  $\alpha$ -model is similar to the  $p$ -model except that it does not conserve energy in a cascade splitting. From the splitting function (2), its b.g.f. is found to be

$$\begin{aligned} Q[\lambda_0, \lambda_1] = & \frac{1}{2}(\lambda_0 + \lambda_1) \ln(1 - \beta^2) \\ & + \ln \left\{ \cosh \left[ \frac{1}{2} \lambda_0 \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right] \right\} \\ & + \ln \left\{ \cosh \left[ \frac{1}{2} \lambda_1 \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right] \right\} \quad , \end{aligned} \quad (61)$$

clearly different from the b.g.f. (52) for the  $p$ -model. Fig. 2(b) depicts the b.g.f. (61). Note that for  $\lambda_0 = 0$  or  $\lambda_1 = 0$  the two expressions (52) and (61) become identical; consequently, the one-point branching moments of the  $\alpha$ -model are identical to those of the  $p$ -model given in (53), (57) and (58):

$$\left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \Big|_{\lambda=0} \right)_{\alpha\text{-model}} = \left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \Big|_{\lambda=0} \right)_{p\text{-model}} . \quad (62)$$

This is the reason why, in a multifractal approach, the two models look the same asymptotically. To see differences between them, one must consider two-point branching moments: for the  $\alpha$ -model they all vanish,

$$\frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n-n_1}} \Big|_{\lambda=0} = \langle (\ln q_0)^{n_1} (\ln q_1)^{n-n_1} \rangle_c = 0 \quad , \quad (63)$$

where  $1 \leq n_1 < n$ . We hence see that the two-point branching moments are sensitive to the violation of energy conservation in the splitting function. As a consequence of (62) and (63), the cumulant densities  $C_{\kappa_1, \dots, \kappa_n}$  of the  $\alpha$ - and  $p$ -model now look slightly different. Fig. 4(a) compares  $C_{\kappa_1, \kappa_2}$  of second order as a function of the ultrametric distance (27).

## C. SRST cascade

The  $p$ -model is able to describe the multifractal aspects of the intermittent fluctuations occurring in the energy dissipation field in fully developed turbulence. However, due to its simplicity, the experimental multiplier distributions cannot be reproduced. In Ref. [13], a modification of the  $p$ -model was proposed which accounts for the correct multiplier distributions; we call this modification the SRST cascade model.

Insertion of its splitting function (3) into Eq. (22) yields an analytic expression for its b.g.f.,

$$Q[\lambda_0, \lambda_1] = \ln \left[ \int_0^1 dz z^{\lambda_0 + \beta - 1} (1 - z)^{\lambda_1 + \beta - 1} \right] + \ln \left[ \frac{\Gamma(2\beta)}{\Gamma(\beta)^2} \right] + (\lambda_0 + \lambda_1) \ln 2 \quad . \quad (64)$$

Since  $\beta = 3.2$  and  $\lambda_0 \approx 0 \approx \lambda_1$ , so that  $\lambda_0 + \beta > 0$  and  $\lambda_1 + \beta > 0$ , the integral appearing in the first term of the right hand side can be identified with the beta function  $B(\lambda_0 + \beta, \lambda_1 + \beta) = \Gamma(\lambda_0 + \beta)\Gamma(\lambda_1 + \beta)/\Gamma(\lambda_0 + \lambda_1 + 2\beta)$ . This leads to

$$Q[\lambda_0, \lambda_1] = (\lambda_0 + \lambda_1) \ln 2 + \ln \left( \frac{\Gamma(2\beta)}{\Gamma(\lambda_0 + \lambda_1 + 2\beta)} \frac{\Gamma(\lambda_0 + \beta)}{\Gamma(\beta)} \frac{\Gamma(\lambda_1 + \beta)}{\Gamma(\beta)} \right) \quad . \quad (65)$$

This result is illustrated in Fig. 2(c).

From (65) the spatial cumulant densities  $C_{\kappa_1, \dots, \kappa_n}$  can be calculated in the straightforward manner presented in Sec. III E. In the lowest even orders, the results are very similar to those obtained for the  $p$ -model; the odd orders, however, are not equal to zero anymore.

Making use of the relationship (45) the result (65) translates into the following multifractal exponents:

$$\tau(\nu) = 1 + \frac{1}{\ln 2} \ln \left( \frac{\Gamma(2\beta)}{\Gamma(\nu + 2\beta)} \frac{\Gamma(\nu + \beta)}{\Gamma(\beta)} \right) \quad . \quad (66)$$

Note that this expression, which is illustrated in Fig. 5, is only defined for  $\nu + \beta > 0$ ; for  $\nu + \beta \leq 0$  the integral in (64) diverges. This is equivalent to the statement that the negative moments with  $\nu \leq -\beta$  of the splitting functions (3) and (4) do not exist. For the multifractal spectrum to exist over the full  $\nu$  range, however, moments of all orders, both positive and negative, must be finite. The absence of finite negative moments hence implies that it is not possible to construct full  $\tau(\nu)$  and  $f(\alpha)$  curves for this specific splitting function. It is thus an example of a well-defined selfsimilar cascade process which cannot be described fully by the multifractal formalism.

#### D. SRST cascade with no energy conservation

Experimentally, the intermittent structures in the three-dimensional energy dissipation field of fully developed turbulence are observed on a one-dimensional cut. Although energy is conserved in three dimensions, this is probably not the case in one dimension. For the multiplicative branching models, this has the consequence that the splitting function  $p(q_0, q_1)$  cannot be expected to conserve energy. In this spirit, the expression (4) represents an untested extrapolation of the SRST multiplier distribution (3), which has been deduced from one-dimensional data under the assumption of energy conservation [13].

Insertion of the splitting function (4) into Eq. (22) yields the corresponding branching generating function:

$$Q[\lambda_0, \lambda_1] = (\lambda_0 + \lambda_1) \ln 2 + \ln \left( \frac{\Gamma(2\beta)}{\Gamma(\lambda_0 + 2\beta)} \frac{\Gamma(\lambda_0 + \beta)}{\Gamma(\beta)} \right) + \ln \left( \frac{\Gamma(2\beta)}{\Gamma(\lambda_1 + 2\beta)} \frac{\Gamma(\lambda_1 + \beta)}{\Gamma(\beta)} \right) ; \quad (67)$$

the explicit derivation is analogous to the one given in Sec. VC. The illustration of (67) is given in Fig. 2(d); it differs from the branching generating function (65) of the SRST cascade model with energy conservation. As in the  $p/\alpha$ -model comparison, the one-point derivatives

$$\left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \Big|_{\lambda=0} \right)_{\text{SRST (noEC)}} = \left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n} \Big|_{\lambda=0} \right)_{\text{SRST (EC)}} \quad (68)$$

of the SRST cascade model with (EC) and without (no EC) energy conservation are identical, while for the two-point derivatives we find ( $1 \leq n_1 < n$ )

$$\left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n-n_1}} \Big|_{\lambda=0} \right)_{\text{SRST (noEC)}} = 0 \neq \left( \frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n-n_1}} \Big|_{\lambda=0} \right)_{\text{SRST (EC)}} . \quad (69)$$

Consequently, the cumulant densities  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}$  of the SRST-cascade model with and without energy conservation are different.

## VI. CONCLUSIONS

With a clever change of variables from energy densities  $\epsilon_{\mathbf{\kappa}}$  to the singularity strengths  $\alpha_{\mathbf{\kappa}}$  or  $\ln \epsilon_{\mathbf{\kappa}}$ , we have derived an analytic expression for the multivariate generating function of binary multiplicative cascade models. The latter completely describes the  $n$ -point statistics, i.e. the spatial (cumulant) correlation densities of arbitrary order  $n$ . The key input has been a bivariate branching generating function, which is related to the underlying splitting function of the binary multiplicative cascade process via a two-dimensional Laplace transform. This branching generating function can be understood as a natural and, for selfsimilar binary cascade processes, complete generalisation of the multifractal mass exponents. While its properties completely fix the spatial correlation densities, the multifractal mass exponents do not. Various cascade models, relevant to fully developed turbulence, have been discussed to underpin this point.

We have shown that, given that the experimentally measurable cumulants in  $(\ln \epsilon_{k_1 \dots k_J}^{(J)})$  are  $n$ -fold derivatives of the branching generating function, the latter can in principle be reconstructed from the former. With the help of Eqs. (48) and (49), the b.g.f. can then be inverted into the splitting function via a two-dimensional inverse Laplace transform. In this way, the violation of energy conservation along one-dimensional cuts through the three-dimensional energy dissipation field can be inspected.

Before this new approach is applied directly to (turbulence) data, however, a number of complications will have to be dealt with. In order to infer the branching generating function from the cumulant densities  $C_{\mathbf{\kappa}_1, \dots, \mathbf{\kappa}_n}$  (to all



orders, in principle), a very effective representation of the latter has to be found; here, as in Refs. [10,11], a wavelet transformation might be useful to compress the information contained in the cumulant densities.

Moreover, there is the problem of non-homogeneity: as a consequence of the hierarchical nature of the cascade evolution, the theoretical correlation functions are not invariant with respect to spatial translations, in contradiction to experimental measurements. It remains to be seen whether and in what way a scheme to restore homogeneity, as for example the one used in [16], influences or destroys the capability of inferring the branching generating function.

Furthermore, the observed statistical dependence of multipliers [13,17,18] has to be taken into account. Recent simulations [19] indicate that this issue is closely linked to the restoration of homogeneity. This is in agreement with the conclusions reached by Nelkin and Stolovitzky [18] by a different route, who argue that the experimentally proven dependence of multiplier distributions on the position of the subinterval implies that the multipliers are not statistically independent. Since any scheme to restore homogeneity will necessarily average out subinterval positions in some way, it will likely influence the multipliers' statistical dependence also. This remains to be explored in detail [19].

Finally, the binary structure of the selfsimilar cascade processes discussed in this paper may not be appropriate: assuming that the physical processes themselves are, indeed, selfsimilar, the best selfsimilar basis for a scaling analysis (such as a specific wavelet) should be selected by the data itself.

Once these points are clarified, new information can hopefully be gleaned from the analysis of “fully developed turbulence data”. Besides fully-developed turbulence, we envisage many and diverse applications of our analytic solution in other branches of physics. The case of QCD branching processes immediately comes to mind. For the latter, the  $\alpha$ - and  $p$ -models have already been used in this context as simulation toy models [20,21]. Implications in this and, for example, random multiplicative process calculations in large-scale structure formation in the universe [22] remain to be explored.

## ACKNOWLEDGMENTS

This work was supported in part by the South African Foundation for Research Development. PL acknowledges support by APART of the Austrian Academy of Sciences.

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